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Lattice theta constants vs Riemann theta constants and NSR superstring measures

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Abstract: We discuss relations between two different representations of hypothetical holomorphic NSR measures, based on two different ways of constructing the semi-modular forms of weight 8. One of these ways is to build forms from the ordinary Riemann theta constants and another — from the lattice theta constants. We discuss unexpectedly elegant relations between lattice theta constants, corresponding to 16-dimensional self-dual lattices, and Riemann theta constants and present explicit formulae expressing the former ones through the latter. Starting from genus 5 the modular-form approach to construction of NSR measures is clearly sick and it seems to fail completely already at genus 6.

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1 Introduction

After the famous work of A.Belavin and V.Knizhnik [1] it turned out that perturbative string theory can by built using the Mumford measure [2] on the moduli space of algebraic curves, that is summation over all world-sheets can be replaced by integration over the moduli space with this measure. For NSR superstring there should be a whole collection of measures for every genus [3], corresponding to different boundary conditions for fermionic fields. They are labeled by semi-integer theta-characteristic: collections of zeroes and unities, associated with non-contractable cycles of the Riemann surface. While bosonic string could be studied without Belavin-Knizhnik theorem, for super- and heterotic strings this is absolutely impossible, because GSO projection requires to sum holomorphic NSR measures over characteristics before taking their bilinear combinations with complex conjugate measures.

It is a long-standing problem to find these NSR superstring measures (see [4] for a recent review and numerous references). The direct way to derive them from the first principles turned to be very hard, and it took around 20 years before E.D'Hoker and D.Phong in a long series of papers managed to do this in the case of genus 2 [5]. Higher genera seem to be even harder. However, there is another, seemingly simpler approach to the problem: to *guess* the answer from known physical and mathematical requirements which it is supposed to satisfy. This approach proved to be quite effective in the case of bosonic strings [6] and, after the new insight from [5] it was successfully applied to the case of NSR measures in [7]–[13]. In the present paper we discuss ansätze for NSR measures, which were proposed following this second way of attacking the problem.

In [6] the physical problem *for low genera* was reformulated as one in the theory of modular forms: one needs to construct semi-modular forms of the weight 8 with certain properties. There are two obvious ways to represent such forms: through *ordinary* Riemann theta-constants and through *lattice* theta-constants. Accordingly there are two ansätze for NSR measures at low genera. One — in terms of Riemann theta constants – was suggested in [5, 7] and in its final appealing form by S. Grushevsky in [8] and the other — in terms of lattice theta constants, associated with the eight (six odd and two even) 16-dimensional unimodular lattices. – by M.Oura, C.Poor, R.Salvati Manni and D.Yuen in [13].

In fact the theory of modular forms is hard, it is being built anew for every new genus, and it is even a question which benefits more from the other: string theory from the theory of modular forms or vice versa. In this paper we continue the study of relations between lattice theta constants and Riemann theta constants. We write explicit formulae expressing the lattice theta constants through Riemann theta constants in all genera and for all odd lattices except one. For this remaining one we write explicit formulae in genera $g \leq 4$ and discuss a hypothesis of how it may look in higher genera. These formulae look as follows:

$$\vartheta_p^{(g)} = 2^{-gp}\xi_p^{(g)}, \quad p = 0..4, \tag{1.1}$$

$$\vartheta_5^{(g)} = 2^{-\frac{g(g-1)}{2}} \cdot \left(\prod_{i=1}^{g}(2^i - 1)\right)^{-1} G_g^{(g)} \tag{1.2}$$

(see section 3 for description of our notation). It is interesting that for the first five odd lattices the proof is provided by a simple rotation of the lattice with the help of the so-called Hadamard matrices.

In other parts of the paper we discuss modular-form ansätze for NSR measures and relations between them. According to [1, 6] the measures are written as

$$d\mu[e] = \Xi[e]d\mu, \tag{1.3}$$

where $d\mu$ is the Mumford measure, and they are subjected to the following conditions:

1. $\Xi[e]$ are (semi-)modular forms of weight 8, at least for low genera,
2. they satisfy factorization property when Riemann surface degenerates,
3. the "cosmological constant" should vanish: $\sum_e \Xi[e] = 0$,
4. for genus 1 the well known answer should be reproduced.

For detailed description of these properties see the next section.

Let us briefly describe the modern ansätze [7]–[13], inspired by [5].

In Grushevsky ansatz $\Xi[e]$ is written as a linear combination of functions $\xi_p[e]$, which are sums over sets of p characteristics of monomials in Jacobi theta constants of order 16. This is literally true for genera $g \leq 4$, for $g > 4$ roots of monomials in Riemann theta constants appear, but the total degree remains 16. This ansatz may in principle be written for any genus, however it satisfies all conditions only for genera $g \leq 4$. At genus 5 in its pure form it fails to satisfy the cosmological constant property [12].

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In OPSMY ansatz expressions for $\Xi[0]$ are written in terms of lattice theta constants, corresponding to six odd unimodular lattices of dimension 16. OPSMY showed that for $g \leq 4$ these lattice theta constants span the entire space of holomorphic functions on the Siegel half-space $\mathcal{H}$, which are modular forms under $\Gamma_g(1,2)$. However, because the number of these lattices is fixed, they doubtly span this space for $g \geq 6$, because the dimension of this space can grow (in fact, it is known that it doesn't grow starting from $g \geq 17$; in principle it is possible that this dimension becomes stable already at genus 5). This fact that the number of lattice theta constants is limited leads to the following: OPSMY ansatz cannot be written for $g \geq 6$ because in these cases one cannot compose from lattice theta constants anything that would satisfy factorization constraint.

Both Grushevsky and OPSMY ansätse potentially have problems in genus 5, because it turned out that in their pure form the cosmological constant does not vanish. Grushevsky and OPSMY then propose to resolve this problem by brute force: if we have a sum of several terms, which should be zero, but apparently is not,

$$\sum_e \Xi[e]^{(5)}(\tau) = F^{(5)}(\tau) \neq 0, \tag{1.4}$$

then we can subtract from each term the sum, divided by number of terms. Then the sum of the modified terms will obviously vanish:

$$\sum_e \widetilde{\Xi}[e]^{(5)}(\tau) = \sum_e \left(\Xi[e]^{(5)}(\tau) - \frac{1}{N_{\text{even}}} F^{(5)}(\tau) \right) = 0 \tag{1.5}$$

This indeed solves the problem in genus 5, because in this particular case such change of $\Xi[e]$ does not spoil the factorization constraint, since F for all genera $g < 5$ vanishes on moduli subspace in the Siegel half-space and thus vanishes whenever the genus-5 surface degenerates. Thus at genus 5 there can be a way out of cosmological constant problem, at least formally.

However this solution for genus 5 doesn't seem very natural because of the following consideration. For all genera $g \leq 3$ the moduli space (i.e. the Jacobian locus) coincides with the Siegel half-space. Therefore the cosmological constant, being zero on moduli, vanishes on entire Siegel space. However, for genus 4 the moduli space becomes non-trivial: it is a rather sophisticated subspace of codimension one in the Siegel half-space. And in this case cosmological constant vanishes only on moduli, but remains nonzero at other points of the Siegel half-space. It is more than natural to expect the same for $g = 5$: that cosmological constant also does not vanish on the entire Siegel half-space in genus 5. However the ansatz, obtained by subtraction trick makes cosmological constant zero on *entire* $\mathcal{H}$. Already this makes it somewhat unnatural. This feeling is confirmed by the fact that the trick fails to work for genus 6 and higher: if we try to resolve the problem with cosmological constant in the same way, we lose the factorization property already at $g = 6$.

It deserves emphasizing that there is *no* reason to believe that $\Xi[e]$ in (1.3) is expressed through modular forms for $g > 4$. One could only hope and try. As we see, the result seems negative, still a lot of interesting details can be learnt in the process. Anyhow, at this moment it is unclear what to do with the superstring measures at genus 6 and above (and the suggested answer for $g = 5$ is also not very convincing).

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The structure of the paper is as follows. In section 2 we discuss mathematical conditions on NSR superstring measures. In section 3 Riemann theta constants are introduced and Grushevsky ansatz is described. In the next section 4 lattice theta constants, associated with self-dual 16-dimensional lattices, are introduced and OPSMY ansatz is discussed. Section 5 is central to the paper and is devoted to explicit relation between lattice and ordinary theta constants. The following section 6 is about the strange behaviour of the theta constant, associated with the sixth odd lattice $(D_8 \oplus D_8)^+$. And finally in the last section 7 we discuss the relation between Grushevsky and OPSMY ansätze. We show *explicitly* that Grushevsky and OPSMY ansätze coincide for genera $g \leq 4$, which is in perfect agreement with the uniqueness properties proved by both Grushevsky and OPSMY. Then we discuss the relation between the ansätze for genus 5.

2 The problem of finding superstring measures

In this section we discuss the mathematical problem of finding the NSR superstring measures in the framework of the modular-form hypothesis [6].

First of all, we review the setting. Superstring measures at genus g, which we are intended to find, form a set of measures $\{d\mu_e\}$ on the moduli space of algebraic curves of genus g. Here index e stands for an *even characteristic*, labeling boundary conditions for fermionic fields on the curve [3]. A *characteristic* is by definition a collection of two g-dimensional $\mathbb{Z}_2$-vectors, i.e. $e \in \left(\mathbb{Z}_2^g\right)^2$. We write

$$e = \begin{bmatrix} \vec{\delta} \\ \vec{\varepsilon} \end{bmatrix}, \tag{2.1}$$

where $\vec{\delta}, \vec{\varepsilon} \in \mathbb{Z}_2^g$. A characteristic is called *even* if the scalar product $\vec{\delta} \cdot \vec{\varepsilon}$ is even. There are $N_{\text{even}} = 2^{g-1}\left(2^g + 1\right)$ even characteristics in genus g.

On the moduli space there is already a distinguished measure – the *Mumford measure* $d\mu$ [1, 2, 6].

It was proposed in [1, 6] that superstring measures are expressed in terms of the Mumford measure in the following way:

$$d\mu_e = \Xi_e d\mu, \tag{2.2}$$

with Ξ_e being some functions on moduli space, satisfying certain conditions.

To pass to this conditions we shall first introduce a convenient way to describe *some* functions on moduli space. Period matrices of the curves, corresponding to particular points in the moduli space, lie in the so-called *Siegel half-space* $\mathcal{H}^{(g)}$. $\mathcal{H}^{(g)}$ is simply the space of $g \times g$ symmetric complex matrices τ with positive definite imaginary part and has (complex) dimension $\frac{g(g+1)}{2}$. Period matrices form the so-called *Jacobian locus* $\mathcal{M}$ inside it, which is a submanifold (strictly speaking, sub*orbi*fold) of complex dimension $3g - 3$ (for $g \geq 2$; for $g = 1$ the dimension is also 1). We then can identify the moduli space with the Jacobian locus and consider a class of functions on moduli space, which depend on τ, i.e. which can be somehow analytically continued to $\mathcal{H}^{(g)}$. One should be cautious

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here: not all the functions on moduli space are of this type, most important, the ones that arise in the free-field calculus on complex curves [14], and are the building blocks for the string measures in straightforward approach, do not usually belong to this class. Still, at least at low genera, they combine nicely into functions, which depend only on τ (the first *non-trivial* example of this kind was the formula for the Mumford measure at genus 4 in [6] — actually not proved in any alternative way till these days), and this motivated the search for $\Xi[e]$ inside this class [15] and nearby [16].

Also let us define the notion of *modular form*, since under above hypothesis $\Xi[e]$ would be of such type. The modular group is defined as $\Gamma^{(g)} =^{\text{def}} \mathrm{Sp}(2g, \mathbb{Z})$ and it acts on $\mathcal{H}^{(g)}$ as follows:

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \geq g, \tag{2.3}$$

$$\gamma : \tau \mapsto (A\tau + B)(C\tau + D)^{-1}. \tag{2.4}$$

Then holomorphic function f on $\mathcal{H}$ is called a *modular form of weight k with respect to subgroup Γ' of the modular group* if it satisfies the following property:

$$f(\gamma\tau) = \det(C\tau + D)^k f(\tau) \tag{2.5}$$

for all $\gamma \in \Gamma'$.

We will be interested in one particular subgroup of the modular group, which is called $\Gamma(1, 2)$. It is defined as follows: an element

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.6}$$

of the modular group belongs to $\Gamma(1, 2)$ iff all elements on diagonals of matrices AB^T and CD^T are even. This subgroup is interesting in conjunction with theta constants, which will be defined in following sections. Now we can say that while action of a general element of the modular group on a Riemann theta constant changes its characteristic, the action of an element of $\Gamma(1, 2)$ leaves zero characteristic invariant.

We finally come to the announced conditions on functions Ξ_e and are eventually able to formulate the problem about superstring measures as a well-posed mathematical problem.

So, the question is as follows: are there any functions Ξ_e in every genus g, which satisfy the following three properties?

1. Ξ_e is a modular form of weight 8 with respect to subgroup $\Gamma_e \subset \Gamma \geq g$, at least when restricted to the Jacobian locus. Sometime such forms are called semi-modular. The subgroup is defined as $\Gamma_e =^{\text{def}} \gamma[e]\Gamma_g(1, 2)\gamma[e]^{-1}$, where $\gamma[e]$ is an element of Γ_g, which transforms the zero characteristic to characteristic e. Saying that an element of the modular group acts on a characteristic, we mean that it acts on the *Riemann theta constant*, associated with this characteristic and transforms it into Riemann theta constant with another characteristic. Riemann theta constants will be defined in section 3.

2. Ξ_e satisfies the following *factorization property*:

$$\Xi^{(g)}[e]\begin{pmatrix}\tau^{(g_1)} & 0\\ 0 & \tau^{(g-g_1)}\end{pmatrix} = \Xi^{(g_1)}_{e_1}\left(\tau^{(g_1)}\right)\,\Xi^{(g-g_1)}_{e/e_1}\left(\tau^{(g-g_1)}\right) \tag{2.7}$$

3. $\Xi[e]$ satisfies the *property of vanishing cosmological constant*:

$$\sum_e \Xi_e\left(\tau\right) = 0 \tag{2.8}$$

   One should keep in mind that the naming convention for this property is a bit abused, because cosmological constant is actually the integral of the total measure over all string configurations. Here it is *not* the case: when we say that cosmological constant vanishes, we mean that this happens point-wise: the total measure is zero at *every point* τ of the moduli space.

4. For genus one Ξ_e reproduces the known answer from elementary superstring theory [3]:

$$\Xi_e \geq 1 = \theta^4[e]\prod_{e'}^{3}\theta[e']^4 = \theta^{16}[e] - \frac{1}{2}\theta^8[e]\left(\sum_{e'}\theta^8[e']\right) \tag{2.9}$$

   Here $\theta[e]\left(\tau\right)$ stands for Jacobi theta constant, see section 3 below. This property is important, because factorization condition iteratively reduces all measures to genus one.

Addressing this mathematical problem is actually an attempt to guess the answer for superstring measures from their known properties instead of doing the calculation from the first principles, which is very difficult in higher genera. We would face a difficulty on this way if it turned out that there are several different possible collections Ξ_e in some genus, satisfying all these conditions. However, this is not the case, moreover the situation is quite the opposite. There are two ansätze proposed, which give the same results up to the genus 5 and do not work further. Thus today it looks like it is impossible to give the answer for genera $g \geq 6$ in such terms. The answer to the overoptimistic question, if superstring measures or, more carefully, the ratios $d\mu[e]/d\mu$, can be represented as semi-modular forms on entire Siegel half-space for all genera, now seems to be negative. We should look either for more sophisticated combinations of modular forms, including residues of Schottky forms like in bosonic measure at genus four, or even switch to the functions, which are modular forms only when restricted to Jacobian locus.

In the two following sections we review the two available ansätze and then use them to illustrate this negative claim.

3 Grushevsky ansatz

First of all, we need to introduce the *Riemann theta constants*, which are the functions on the Siegel half-space, in terms of which the ansatz is written. Note that this usual terminology is rather misleading here, because they are indeed functions, not constants on the Siegel half-space. This naming convention reflects the fact that there is a notion of *"Riemann theta function"* which is a function not only of the modular parameter τ, but also of coordinates $\vec{z}$ on the Jacobian (a g-dimensional torus), associated to this value of modular parameter. In this context the theta functions with $\vec{z}=0$ are usually called "theta constants", because they do not depend on $\vec{z}$. In the present paper we will use only theta constants – and call them *constants* to avoid possible confusion, despite we need and use them as *functions* on the Siegel half-space.

So the Riemann theta constant with characteristic at genus g is defined as follows [17]:

$$\theta\begin{bmatrix}\vec{\delta}\\ \vec{\varepsilon}\end{bmatrix}(\tau) \stackrel{\text{def}}{=} \sum_{\vec{n}\in\mathbb{Z}^g} \exp\left(\pi i\left(\vec{n}+\vec{\delta}/2\right)^T \tau\left(\vec{n}+\vec{\delta}/2\right) + \pi i\left(\vec{n}+\vec{\delta}/2\right)^T \vec{\varepsilon}\right) \tag{3.1}$$

All Riemann theta constants with odd characteristics are identically zero.

It turns out that Riemann theta constants are the nice building blocks for modular forms on the Siegel half-space. More precisely, they behave in the following way under modular transformations $\gamma\in\Gamma(1,2)$:

$$\begin{aligned}\theta\begin{bmatrix}D\vec{\delta}-C\vec{\varepsilon}\\ -B\vec{\delta}+A\vec{\varepsilon}\end{bmatrix}(\gamma\tau) = \zeta_\gamma \det\left(C\tau+D\right)^{1/2}&\\ \times\exp\left(\frac{\pi i}{4}\left(2\vec{\delta}^T B^T C\vec{\varepsilon}-\vec{\delta}^T B^T D\vec{\delta}-\vec{\varepsilon}^T A^T C\vec{\varepsilon}\right)\right)\theta\begin{bmatrix}\vec{\delta}\\ \vec{\varepsilon}\end{bmatrix}(\tau)\,,&\end{aligned} \tag{3.2}$$

where ζ_γ is some eighth root of unity which depends only on γ. It is then straightforward to see that $\theta^{16}[0]$ is a modular form of weight 8 w.r.t. $\Gamma(1,2)$. The sum of $\theta^{16}[e]$ over all even characteristics will be a modular form of weight 8 w.r.t. the whole modular group. When we write [0] we everywhere mean zero characteristic, i.e. characteristic, for which all elements of both vectors $\vec{\delta}$ and $\vec{\varepsilon}$ are zeroes.

In any genus Grushevsky ansatz can be expressed through the following combinations of Riemann theta constants [4, 7] (for brevity we write characteristics as indices in the r.h.s.)

$$\begin{aligned}
&\xi_0^{(g)}[e] = \theta_e^{16},\\
&\xi_1^{(g)}[e] = \theta_e^{8}\sum_{e_1}^{N_e}\theta_{e+e_1}^{8},\\
&\xi_2^{(g)}[e] = \theta_e^{4}\sum_{e_1,e_2}^{N_e}\theta_{e+e_1}^{4}\theta_{e+e_2}^{4}\theta_{e+e_1+e_2}^{4},\\
&\xi_3^{(g)}[e] = \theta_e^{2}\sum_{e_1,e_2,e_3}^{N_e}\theta_{e+e_1}^{2}\theta_{e+e_2}^{2}\theta_{e+e_3}^{2}\theta_{e+e_1+e_2}^{2}\theta_{e+e_1+e_3}^{2}\theta_{e+e_2+e_3}^{2}\theta_{e+e_1+e_2+e_3}^{2},\\
&\quad\dots\\
&\xi_p^{(g)}[e] = \sum_{e_1,\dots,e_p}^{N_e}\left\{\theta_e\cdot\left(\prod_i^p\theta_{e+e_i}\right)\cdot\left(\prod_{i<j}^p\theta_{e+e_i+e_j}\right)\cdot\left(\prod_{i<j<k}^p\theta_{e+e_i+e_j+e_k}\right)\cdot\ldots\cdot\theta_{e+e_1+\ldots+e_p}\right\}^{2^{4-p}}
\end{aligned} \tag{3.3}$$

in the following way:

$$\Xi^{(g)}[0] = \sum_{p=0}^{g} h_p^{(g)} \xi_p^{(g)}[0], \tag{3.4}$$

where

$$h_p^{(g)} = (-1)^p \cdot 2^{\frac{(g-p)^2-(g+p)}{2}} \cdot \left(\prod_{i=1}^{p} (2^i - 1) \prod_{i=1}^{g-p} (2^i - 1) \right)^{-1} \tag{3.5}$$

Other $\Xi[e]$ are obtained just by substituting e instead of 0 into $\xi[0]$. Formula (3.5) becomes more compact if written in terms of the so-called 2-factorial numbers (for definition of q-factorials and q-binomial coefficients see, for example, [18]):

$$h_p^{(g)} = \frac{(-1)^p}{[p]_2![g-p]_2!} \cdot 2^{\frac{(g-p)^2-(g+p)}{2}} \tag{3.6}$$

Explicitly the first five forms look like (when we write ξ_p without characteristic, we mean $\xi_p[0]$)

$$\Xi^{(1)}[0] = \xi_0 - \frac{1}{2}\xi_1, \tag{3.7}$$

$$\Xi^{(2)}[0] = \frac{2}{3}\xi_0 - \frac{1}{2}\xi_1 + \frac{1}{12}\xi_2, \tag{3.8}$$

$$\Xi^{(3)}[0] = \frac{8}{21}\xi_0 - \frac{1}{3}\xi_1 + \frac{1}{12}\xi_2 - \frac{1}{168}\xi_3, \tag{3.9}$$

$$\Xi^{(4)}[0] = \frac{64}{315}\xi_0 - \frac{4}{21}\xi_1 + \frac{1}{18}\xi_2 - \frac{1}{168}\xi_3 + \frac{1}{5040}\xi_4, \tag{3.10}$$

$$\Xi^{(5)}[0] = \frac{1024}{9765}\xi_0 - \frac{32}{315}\xi_1 + \frac{2}{63}\xi_2 - \frac{1}{252}\xi_3 + \frac{1}{5040}\xi_4 - \frac{1}{312480}\xi_5 \tag{3.11}$$

Alternatively one can use the so-called Grushevsky basis [4, 8]:

$$\begin{aligned}
G_0^{(g)}[e] &= \theta_e^{16}, \\
G_1^{(g)}[e] &= \theta_e^8 \sum_{e_1 \neq 0}^{N_e} \theta_{e+e_1}^8, \\
G_2^{(g)}[e] &= \theta_e^4 \sum_{e_1 \neq e_2 \neq 0}^{N_e} \theta_{e+e_1}^4 \theta_{e+e_2}^4 \theta_{e+e_1+e_2}^4, \\
G_3^{(g)}[e] &= \theta_e^2 \sum_{e_1 \neq e_2 \neq e_3 \neq 0}^{N_e} \theta_{e+e_1}^2 \theta_{e+e_2}^2 \theta_{e+e_3}^2 \theta_{e+e_1+e_2}^2 \theta_{e+e_1+e_3}^2 \theta_{e+e_2+e_3}^2 \theta_{e+e_1+e_2+e_3}^2, \\
&\dots \\
G_p^{(g)}[e] &= \sum_{e_1 \neq \ldots \neq e_p \neq 0}^{N_e} \left\{ \theta_e \cdot \left(\prod_{i}^{p} \theta_{e+e_i} \right) \cdot \left(\prod_{i<j}^{p} \theta_{e+e_i+e_j} \right) \right. \\
&\qquad \left. \cdot \left(\prod_{i<j<k}^{p} \theta_{e+e_i+e_j+e_k} \right) \cdot \ldots \cdot \theta_{e+e_1+\ldots+e_p} \right\}^{2^{4-p}}
\end{aligned} \tag{3.12}$$

Here sums are taken over sets of characteristics in which all characteristics are different. This basis is related to $\xi_p^{(g)}$ basis as follows:

$$G_p^{(g)}[e] = \sum_{k=0}^{p} (-1)^{k+p} \cdot 2^{\frac{(p-k)(p-k-1)}{2}} \cdot \frac{\prod_{i=1}^{p} (2^i - 1)}{\prod_{i=1}^{k} (2^i - 1) \prod_{i=1}^{p-k} (2^i - 1)} \xi_k^{(g)}[e] =$$

$$= \sum_{k=0}^{p} (-1)^{k+p} \cdot 2^{\frac{(p-k)(p-k-1)}{2}} \binom{p}{k}_2 \xi_k^{(g)}[e], \tag{3.13}$$

where in the last line the 2-binomial coefficients were used (also known as Gaussian binomial coefficients for $q = 2$).

Expressions for $\Xi^{(g)}[0]$ are then written in terms of Grushevsky basis as

$$\Xi^{(g)}[0] = \sum_{p=0}^{g} f_p^{(g)} G_p^{(g)}[0], \tag{3.14}$$

where

$$f_p^{(g)} = (-1)^p \cdot 2^{-g} \cdot \left(\prod_{i=1}^{p} (2^i - 1) \right)^{-1} = \tag{3.15}$$

$$= (-1)^p \cdot 2^{-g} \cdot \frac{1}{[p]_2!}. \tag{3.16}$$

Explicitly the first five lines look like

$$\begin{aligned}
\Xi^{(1)}[0] &= \frac{1}{2} (G_0 - G_1), \\
\Xi^{(2)}[0] &= \frac{1}{4} \left(G_0 - G_1 + \frac{1}{3} G_2 \right), \\
\Xi^{(3)}[0] &= \frac{1}{8} \left(G_0 - G_1 + \frac{1}{3} G_2 - \frac{1}{21} G_3 \right), \\
\Xi^{(4)}[0] &= \frac{1}{16} \left(G_0 - G_1 + \frac{1}{3} G_2 - \frac{1}{21} G_3 + \frac{1}{315} G_4 \right), \\
\Xi^{(5)}[0] &= \frac{1}{32} \left(G_0 - G_1 + \frac{1}{3} G_2 - \frac{1}{21} G_3 + \frac{1}{315} G_4 - \frac{1}{9765} G_5 \right)
\end{aligned} \tag{3.17}$$

When we write G_p without characteristic, we mean $G_p[0]$.

Let us denote the sum of all $\Xi[e]$ of the Grushevsky ansatz, i.e. the proposed cosmological constant, as $F^{(g)}$ in genus g. It can be proved [12] that it is equal (up to a constant factor) to the following celebrated expression in theta constants [1, 6]:

$$F^{(g)} = 2^g \sum_e \theta^{16}[e] - \left(\sum_e \theta^8[e] \right)^2, \tag{3.18}$$

It can be checked that Grushevsky ansatz satisfies the above requirements for superstring measures in genera $g \leq 4$. However it fails to do so in genus 5 because the cosmological constant $F^{(5)}$ turns to be nonzero [12]. Grushevsky proposes a way to overcome this problem: consider instead of $\Xi[e]$ functions $\Xi[e] - N_{\mathrm{even}}^{-1} F$. This obviously solves the cosmological constant problem and moreover does not spoil other properties in the case of genus 5. The factorization property is not spoilt because $F^{(g)}$ is zero on the Jacobian locus for all genera up to 4 includingly, and thus is zero on points of the Jacobian locus of the Siegel space of genus 5 that correspond to factorization.

So, with a bit of modification the ansatz apparently works up to genus 5 includingly. Since the solution looks strange – as we discussed in the introduction – condition (2.8) should probably be modified (strengthened) in order to exclude *such* a way out, however under the present constraints the problem *has* a formal solution for $g = 5$. However for genus 6 and above the same problem persists and can not be cured anymore by the above-mentioned trick even formally, because it spoils the factorization property for that cases: when a genus-six curve degenerates it can become a genus-five curve of generic kind, and $F^{(5)}$ does *not* vanish identically even on the moduli space.

4 OPSMY ansatz

Another ansatz for superstring measures was proposed by M.Oura, C.Poor, R.Salvati Manni and D.Yuen in their paper [13]. It is expressed in terms of *lattice theta constants of 16-dimensional unimodular lattices*, well familiar from the study of string compactifications [19]. Still we need to remind what they are.

An h-dimensional lattice Λ is a subset of $\mathbb{R}^h$ which is spanned by linear combinations with integer coefficients of some h linearly independent vectors. These h vectors together are called the *basis* of the lattice.

Naturally associated with the lattice Λ is a genus-g lattice theta constant: a function on the Siegel space $\mathcal{H}$, defined as

$$\vartheta_\Lambda(\tau) \overset{\mathrm{def}}{=} \sum_{(\vec{p}_1,\ldots,\vec{p}_g)\in\Lambda^g} \exp\left(\pi i(\vec{p}_k \cdot \vec{p}_l)\tau_{kl}\right), \tag{4.1}$$

where $(\vec{p}\cdot\vec{p}')$ denotes the usual Euclidean scalar product of vectors in $\mathbb{R}^h$, and summation is always assumed over repeated indices k and l. Lattice theta-constants have a very simple factorization property: they remain themselves:

$$\vartheta_\Lambda^{(g_1+g_2)}\begin{pmatrix} \tau^{(g_1)} & 0 \\ 0 & \tau^{g_2)} \end{pmatrix} = \vartheta_\Lambda^{(g_1)}\left(\tau^{(g_1)}\right)\vartheta_\Lambda^{(g_2)}\left(\tau^{(g_2)}\right) \tag{4.2}$$

A lattice is called *self-dual*, or *unimodular*, if it coincides with its dual lattice, i.e. if $\Lambda = \Lambda^*$. The dual lattice Λ^* is defined as the set of all vectors $\vec{u}$ of $\mathbb{R}^h$ such that $(\vec{u}\cdot\vec{v}) \in \mathbb{Z}$ *for all* $\vec{v} \in \Lambda$. A lattice is called *even* if Euclidean norms of all basis vectors are even. Otherwise the lattice is called *odd*.

Lattice theta constant corresponding to self-dual h-dimensional lattice with h divisible by 8 is a modular form of weight $h/2$ w.r.t. $\Gamma_g(1,2)$ if the lattice is odd, and w.r.t. the

Short notation for lattice theta constant	Lattice	Gluing vectors
ϑ_0	$\mathbb{Z}^{16}$	—
ϑ_1	$\mathbb{Z}^8 \oplus E_8$	—
ϑ_2	$\mathbb{Z}^4 \oplus D_{12}^+$	$\left(0^4, \frac{1}{2}^{12}\right)$
ϑ_3	$\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+$	$\left(\frac{1}{4}^6, -\frac{3}{4}^2, \frac{1}{4}^6, -\frac{3}{4}^2\right)$
ϑ_4	$\mathbb{Z} \oplus A_{15}^+$	$\left(\frac{1}{4}^{12}, -\frac{3}{4}^4\right), \left(\frac{1}{2}^8, -\frac{1}{2}^8\right), \left(\frac{3}{4}^4, -\frac{1}{4}^{12}\right)$
ϑ_5	$(D_8 \oplus D_8)^+$	$\left(\frac{1}{2}^8, 0^7, 1\right)$
ϑ_6	$E_8 \oplus E_8$	—
ϑ_7	D_{16}^+	$\left(\frac{1}{2}^{16}\right)$

Table 1. Lattice theta constants.

full Γ_g when the lattice is even. Therefore lattice theta constants of 16-dimensional self-dual lattices are semi-modular forms of the weight 8 and of particular interest for building superstring measures.

There are exactly eight 16-dimensional self-dual lattices [20], all of them can be obtained from root lattices of some Lie algebras. We list them with convenient notations in table 1.

The first column lists the naming conventions for theta constants corresponding to lattices, the second column presents lattices themselves and the third column contains *gluing vectors* of particular lattices. Here $\mathbb{Z}^n$ denotes the trivial integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. A_k, D_k and E_k stand for root lattices of the corresponding Lie algebras. Cross over the name of a lattice (i.e. Λ^+) means the union of this lattice with lattices obtained by shifting it by all of its gluing vectors. For notation to be concise, we follow [20] and write a^n for $a, \ldots, a$ with n instances of a (e.g. $\left(\frac{1}{2}^2, -1^2, 0\right)$ means $\left(\frac{1}{2}, \frac{1}{2}, -1, -1, 0\right)$). Also note that $E_8 = D_8^+$ with the gluing vector $\left(\frac{1}{2}^8\right)$. First six lattices in the table are odd and the last two are even.

Note that in the paper [13] the short notations ϑ_i are also used for the lattice theta constants, but our convention for numbering them is *different*. We have a reason for that, which will be explained at the end of section 5.

OPSMY use these lattice theta constants to write ansatz for superstring measures [13]. Their idea is as follows: because all these lattice theta constants are modular forms of weight 8 with respect to subgroup $\Gamma(1,2)$ of the modular group, it is natural to try to build $\Xi[0]$ out of them and then obtain all $\Xi[e]$ by acting on $\Xi[0]$ by modular transformations.

In fact OPSMY prove that up to genus 4 these lattice theta constants span the *entire* space of modular forms of weight 8 with respect to $\Gamma(1,2)$ and also find all the linear relations among them:

$\boldsymbol{g = 1:}$

$$\vartheta_2 = \frac{3}{2}\vartheta_1 - \frac{1}{2}\vartheta_0$$

$$\begin{aligned}\vartheta_3 &= \frac{7}{4}\vartheta_1 - \frac{3}{4}\vartheta_0\\ \vartheta_4 &= \frac{15}{8}\vartheta_1 - \frac{7}{8}\vartheta_0\\ \vartheta_5 &= 2\vartheta_1 - \vartheta_0\end{aligned}\tag{4.3}$$

$\boldsymbol{g = 2:}$

$$\begin{aligned}\vartheta_3 &= \frac{7}{4}\vartheta_2 - \frac{7}{8}\vartheta_1 + \frac{1}{8}\vartheta_0\\ \vartheta_4 &= \frac{35}{16}\vartheta_2 - \frac{45}{32}\vartheta_1 + \frac{7}{32}\vartheta_0\\ \vartheta_5 &= \frac{8}{3}\vartheta_2 - 2\vartheta_1 + \frac{1}{3}\vartheta_0\end{aligned}\tag{4.4}$$

$\boldsymbol{g = 3:}$

$$\begin{aligned}\vartheta_4 &= \frac{15}{8}\vartheta_3 - \frac{35}{32}\vartheta_2 + \frac{15}{64}\vartheta_1 - \frac{1}{64}\vartheta_0\\ \vartheta_5 &= \frac{64}{21}\vartheta_3 - \frac{8}{3}\vartheta_2 + \frac{2}{3}\vartheta_1 - \frac{1}{21}\vartheta_0\end{aligned}\tag{4.5}$$

$\boldsymbol{g = 4:}$

$$\vartheta_5 \underset{\mathcal{M}}{=} \frac{1024}{315}\vartheta_4 - \frac{64}{21}\vartheta_3 + \frac{8}{9}\vartheta_2 - \frac{2}{21}\vartheta_1 + \frac{1}{315}\vartheta_0 \tag{4.6}$$

Here $\mathcal{M}$ under the equation sign means that the equality is valid when restricted to Jacobian locus.

Actually, in the last line a combination $\vartheta_6 - \vartheta_7$ also enters, but it vanishes on Jacobian locus. So the linear relation looks as above only on Jacobian locus. In fact, the combination $\vartheta_6 - \vartheta_7$ would be proportional to the cosmological constant. For $g \geq 5$ all 8 lattice theta constants (including even ones) are linearly independent, and it is unknown if they span the entire space of $\Gamma(1,2)$-modular forms (which is unlikely at least for $g \geq 6$).

Let us now describe the OPSMY ansatz for superstring measures. Our formulas will look a little different from [13]. First, we choose different enumeration of lattice theta constants, as it was already mentioned, and, second, we choose different normalization for the Ξ function for genus 1, so that

$$\Xi^{(1)}[0] = \vartheta_0 - \vartheta_1 = \theta^{16}[0] - \frac{1}{2}\theta^8[0]\sum_e \theta^8[e] \tag{4.7}$$

instead of

$$\Xi^{(1)}_{OPSMY}[0] = \frac{1}{16}\vartheta_0 - \frac{1}{16}\vartheta_1 = \frac{1}{16}\theta^{16}[0] - \frac{1}{32}\theta^8[0]\sum_e \theta^8[e] \tag{4.8}$$

which is chosen in [13]. Therefore $\Xi^{(g)}$ would differ by a 2^{4g} factor.

Let M be the following 6×6 matrix with indexes running from 0 to 5:

$$\begin{aligned}&M_{ij} = 2^{j(1-i)}, && i = 0..4, && j = 1..5,\\ &M_{i0} = 1, && i = 0..5, &&\\ &M_{5j} = 0, && j = 1..5 &&\end{aligned}\tag{4.9}$$

Explicitly,

$$M = \begin{pmatrix} 1 & 2 & 4 & 8 & 16 & 32 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} \\ 1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} & \frac{1}{1024} \\ 1 & \frac{1}{8} & \frac{1}{64} & \frac{1}{512} & \frac{1}{4096} & \frac{1}{32768} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.10}$$

Then $\Xi[0]$ functions of OPSMY ansatz in genera $g \leq 5$ are

$$\Xi^{(g)}[0] = \sum_{k=0}^{5} \left(M^{-1}\right)_{gk} \vartheta_k^{(g)}, \quad g = 1 \dots 5 \tag{4.11}$$

(for genus 5 OPSMY, actually, propose different expression, see below). In this ansatz we cannot write expressions of $\Xi[e]$ for general e as explicitly as in Grushevsky ansatz, $\Xi[e]$ are obtained from $\Xi[0]$ by action of particular element γ_e of the modular group.

Explicitly,

$$\begin{aligned}
\Xi^{(1)}[0] &= \frac{1}{630}\vartheta_0 - \frac{2}{21}\vartheta_1 + \frac{16}{9}\vartheta_2 - \frac{256}{21}\vartheta_3 + \frac{8192}{315}\vartheta_4 - \frac{31}{2}\vartheta_5, \\
\Xi^{(2)}[0] &= -\frac{1}{42}\vartheta_0 + \frac{29}{21}\vartheta_1 - 24\vartheta_2 + \frac{2944}{21}\vartheta_3 - \frac{4096}{21}\vartheta_4 + \frac{155}{2}\vartheta_5, \\
\Xi^{(3)}[0] &= \frac{1}{9}\vartheta_0 - 6\vartheta_1 + \frac{808}{9}\vartheta_2 - 384\vartheta_3 + \frac{4096}{9}\vartheta_4 - 155\vartheta_5, \\
\Xi^{(4)}[0] &= -\frac{4}{21}\vartheta_0 + \frac{184}{21}\vartheta_1 - 96\vartheta_2 + \frac{7424}{21}\vartheta_3 - \frac{8192}{21}\vartheta_4 + 124\vartheta_5, \\
\Xi^{(5)}[0] &= \frac{32}{315}\vartheta_0 - \frac{64}{21}\vartheta_1 + \frac{256}{9}\vartheta_2 - \frac{2048}{21}\vartheta_3 + \frac{32768}{315}\vartheta_4 - 32\vartheta_5
\end{aligned} \tag{4.12}$$

Substituting linear relations from (4.3)–(4.6), we obtain:

$$\Xi^{(1)}[0] = \vartheta_0 - \vartheta_1, \tag{4.13}$$

$$\Xi^{(2)}[0] = \frac{2}{3}\vartheta_0 - 2\vartheta_1 + \frac{4}{3}\vartheta_2, \tag{4.14}$$

$$\Xi^{(3)}[0] = \frac{8}{21}\vartheta_0 - \frac{8}{3}\vartheta_1 + \frac{16}{3}\vartheta_2 - \frac{64}{21}\vartheta_3, \tag{4.15}$$

$$\Xi^{(4)}[0] = \frac{64}{315}\vartheta_0 - \frac{64}{21}\vartheta_1 + \frac{128}{9}\vartheta_2 - \frac{512}{21}\vartheta_3 + \frac{4096}{315}\vartheta_4, \tag{4.16}$$

$$\Xi^{(5)}[0] = \frac{32}{315}\vartheta_0 - \frac{64}{21}\vartheta_1 + \frac{256}{9}\vartheta_2 - \frac{2048}{21}\vartheta_3 + \frac{32768}{315}\vartheta_4 - 32\vartheta_5 \tag{4.17}$$

It turns out that for genus 5 the cosmological constant for this $\Xi^{(5)}$ is still proportional to $\vartheta_6 - \vartheta_7$ and therefore is non-vanishing [12]. Therefore OPSMY propose to cure it in the same way which was used with the Grushevsky ansatz. They find the numerical value of ratio of $\sum_e \Xi[e]$ and $\vartheta_6 - \vartheta_7$ and subtract from $\Xi[0]$ this expression with this coefficient, divided by the number of characteristics. In our normalization this looks like

$$\widetilde{\Xi}^{(5)}[0] = \frac{32}{315}\vartheta_0 - \frac{64}{21}\vartheta_1 + \frac{256}{9}\vartheta_2 - \frac{2048}{21}\vartheta_3 + \frac{32768}{315}\vartheta_4 - 32\vartheta_5 - \frac{686902}{24255}\vartheta_6 + \frac{686902}{24255}\vartheta_7 \tag{4.18}$$

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This ansatz cannot be continued to $g = 6$ and above. One can easily understand the problem, for example, in the following way. The factorization property requires that,

$$\Xi^{(6)}\left(\tau^{(6)}_{1+1+1+1+1+1}\right)=\Xi^{(1)}\left(\tau^{(1)}_1\right)\Xi^{(1)}\left(\tau^{(1)}_2\right)\Xi^{(1)}\left(\tau^{(1)}_3\right)\Xi^{(1)}\left(\tau^{(1)}_4\right)\Xi^{(1)}\left(\tau^{(1)}_5\right)\Xi^{(1)}\left(\tau^{(1)}_6\right), \tag{4.19}$$

and if we had $\Xi^{(6)}$ expressed as a linear combination of lattice theta constants $\Xi^{(6)} = \sum_{p=0}^{7}\alpha_p\vartheta_p$, then, using (4.2),

$$\sum_{p=0}^{7}\alpha_p\vartheta_p^{(1),1}\,\vartheta_p^{(1),2}\,\vartheta_p^{(1),3}\,\vartheta_p^{(1),4}\,\vartheta_p^{(1),5}\,\vartheta_p^{(1),6} = \tag{4.20}$$

$$=\left(\vartheta_0^{(1),1}-\vartheta_1^{(1),1}\right)\left(\vartheta_0^{(1),2}-\vartheta_1^{(1),2}\right)\left(\vartheta_0^{(1),3}-\vartheta_1^{(1),3}\right)\left(\vartheta_0^{(1),4}-\vartheta_1^{(1),4}\right)\left(\vartheta_0^{(1),5}-\vartheta_1^{(1),5}\right)\left(\vartheta_0^{(1),6}-\vartheta_1^{(1),6}\right)$$

For $g = 1$ all theta constants can be expressed in terms of three linearly independent ones ϑ_0, ϑ_1, ϑ_6 with the help of (4.3). If one does this and expands all brackets, one will obtain a system of linear equations on $\{\alpha_i\}$, because after passing to linearly independent functions all coefficients before monomials in them shall vanish if l.h.s. is subtracted from r.h.s. It turns out that in this $g = 6$ case the system of equations simply does not have solutions. Thus the factorization constraint cannot be satisfied. For example, if one does the same things for $g = 5$, one will obtain result (4.17).

5 Lattice theta constants vs Riemann theta constants

In this section we describe the relation between lattice and Riemann theta constants and write down explicit formulae, expressing ones in terms of the others. Namely, we prove that

$$\vartheta_p^{(g)} = 2^{-gp}\xi_p^{(g)}, \quad p = 0\dots 4 \tag{5.1}$$

in any genus. For $p = 0$ and $p = 1$ the statement is rather trivial and already known.

Consider the $p = 2$ case. First, note that the factor of θ_0^4 is common to the l.h.s. and to the r.h.s. of (5.1) in this case, so we divide it out. Then for the right hand side we have

$$\xi_2^{(g)}/\theta_0^4 = \sum_{e_1,e_2}^{N_e}\theta^4_{e_1}\theta^4_{e_2}\theta^4_{e_1+e_2} = \sum_{e_1,e_2}^{N_e}\sum_{\substack{\vec{n}^a_I\in\mathbb{Z}^g,\\ a=1..4,\\ I\in 2^{\{1,2\}}\setminus(\emptyset)}}\exp\left(\pi i\left(\sum_a\left(\vec{n}^a_1+\frac{\vec{\delta}_1}{2}\right)^T\tau\left(\vec{n}^a_1+\frac{\vec{\delta}_1}{2}\right)+\right.\right.$$

$$+\sum_a\left(\vec{n}^a_2+\frac{\vec{\delta}_2}{2}\right)^T\tau\left(\vec{n}^a_2+\frac{\vec{\delta}_2}{2}\right)+\sum_a\left(\vec{n}^a_{12}+\frac{\vec{\delta}_1+\vec{\delta}_2}{2}\right)^T\tau\left(\vec{n}^a_{12}+\frac{\vec{\delta}_1+\vec{\delta}_2}{2}\right)+$$

$$\left.\left.+\left(\sum_a\vec{n}^a_1\right)^T\vec{\varepsilon}_1+\left(\sum_a\vec{n}^a_2\right)^T\vec{\varepsilon}_2+\left(\sum_a\vec{n}^a_{12}\right)^T(\vec{\varepsilon}_1+\vec{\varepsilon}_2)\right)\right) \tag{5.2}$$

We can perform summation over $\vec{\varepsilon}_1$, $\vec{\varepsilon}_2$. Since they take values in $(\mathbb{Z}_2)^g$ and enter the expression as factors of

$$\exp\left(\pi i\left(\sum_a\vec{n}^a_1+\vec{n}^a_{12}\right)^T\vec{\varepsilon}_1\right) \tag{5.3}$$

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and

$$\exp\left(\pi i\left(\sum_a \vec{n}_2^a+\vec{n}_{12}^a\right)^T\vec{\varepsilon}_2\right) \tag{5.4}$$

then all terms, in which at least on element of integer vectors $\vec{v}_1=\sum_a\left(\vec{n}_1^a+\vec{n}_{12}^a\right)$ and $\vec{v}_2=\sum_a\left(\vec{n}_2^a+\vec{n}_{12}^a\right)$ is odd, vanish, and all terms with all these elements being even survive and acquire a factor of 2^{2g}. Therefore, expanding also the sum in $\vec{\delta}$, we obtain

$$\xi_2^{(g)}/\theta_0^4=2^{2g}\sum_{\left(\Lambda_2^+\right)^g}\exp\left(\pi i\sum_{i,j=1}^{g}\left(n_i,n_j\right)\tau_{ij}\right), \tag{5.5}$$

where $\Lambda_2\subset\mathbb{Z}^{12}$ is a 12-dimensional lattice defined as

$$\begin{aligned}\Lambda_2=\Bigg\{&\left(n_1^1,\dots,n_1^4,n_2^1,\dots,n_2^4,n_{12}^1,\dots,n_{12}^4\right)\in\mathbb{Z}^{12}\Bigg|\\ &\left(\sum_a n_1^a+\sum_a n_{12}^a\right)\vdots 2,\ \left(\sum_a n_2^a+\sum_a n_{12}^a\right)\vdots 2\Bigg\}\end{aligned} \tag{5.6}$$

and

$$\Lambda_2^+=\Lambda_2\cup\left(\Lambda_2+\vec{\alpha}\right)\cup\left(\Lambda_2+\vec{\beta}\right)\cup\left(\Lambda_2+\vec{\gamma}\right), \tag{5.7}$$

where

$$\vec{\alpha}=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\ 0,0,0,0,\ \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right), \tag{5.8}$$

$$\vec{\beta}=\left(0,0,0,0,\ \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\ \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right), \tag{5.9}$$

$$\vec{\gamma}=\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\ \frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\ 0,0,0,0\right) \tag{5.10}$$

Therefore, since factors 2^{2g} and 2^{-2g} perfectly cancel each other, to prove the lemma we only need to prove that Λ_2^+ turns to the usual representation of D_{12}^+ under some orthogonal transformation $A:\mathbb{R}^{12}\to\mathbb{R}^{12}$. We now show that this transformation is the following one:

$$A_{12}=\frac{1}{2}\begin{pmatrix}H_4&0&0\\0&H_4&0\\0&0&H_4\end{pmatrix}, \tag{5.11}$$

where H_4 is the following matrix

$$H_4=\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{pmatrix} \tag{5.12}$$

H_4 is a so-called Hadamard matrix, see below for details.

The D_{12}^+ lattice is defined as

$$D_{12}^+ = \left\{ (m_1, \ldots, m_{12}) \in \mathbb{Z}^{12} \cup \left(\mathbb{Z}^{12} + \left(\frac{1}{2}^{12}\right)\right) \Bigg| \left(\sum_k m_k\right) \vdots 2 \right\} \tag{5.13}$$

It is then straightforward to check that A maps every point of Λ_2^+ into point of D_{12}^+ and vice versa. Indeed, Λ_2 consists of vectors (e, e, e) and (o, o, o), where e is either $(0, 0, 0, 0)$ or $(1, 1, 1, 1)$ or one of the $C_2^4 = 6$ vectors like $(1, 1, 0, 0)$, while o is either of the type $(1, 0, 0, 0)$ or $(1, 1, 1, 0)$. All entries are defined modulo 2. So,

$$\Lambda_2 = \Big\{ (e, e, e), (o, o, o) \Big\} \tag{5.14}$$

Similarly

$$D_{12} = \Big\{ (e, e, e), (e, o, o), (o, e, o), (o, o, e) \Big\} \tag{5.15}$$

Now, denote by check a 4-vector shifted by $(1/2, 1/2, 1/2, 1/2)$. Then

$$\Lambda_2^+ = \Big\{ (e, e, e), (\check{e}, \check{e}, e), (\check{e}, e, \check{e}), (e, \check{e}, \check{e}), (o, o, o), (\check{o}, \check{o}, o), (\check{o}, o, \check{o}), (o, \check{o}, \check{o}) \Big\},$$

while

$$D_{12}^+ = \Big\{ (e, e, e), (\check{e}, \check{e}, \check{e}), (e, o, o), (\check{e}, \check{o}, \check{o}), (o, e, o), (\check{o}, \check{e}, \check{o}), (o, o, e), (\check{o}, \check{o}, \check{e}) \Big\}$$

The action of $\frac{1}{2}H_4$, $\left(\frac{1}{2}H_4\right)^2 = I$ is as follows:

$$\begin{array}{ccc}
(0,0,0,0) & \leftrightarrow & (0,0,0,0) \\
(1,1,0,0) & \leftrightarrow & (1,0,1,0) \\
(1,1,1,1) & \leftrightarrow & (2,0,0,0) \\
 & & \\
(1,0,0,0) & \leftrightarrow & (1/2,1/2,1/2,1/2) \\
(1,1,1,0) & \leftrightarrow & (3/2,1/2,1/2,-1/2)
\end{array}$$

$$(-1/2, 1/2, 1/2, 1/2) \leftrightarrow (1/2, -1/2, -1/2, -1/2)$$

what implies that $e \leftrightarrow e$, $o \leftrightarrow \check{e}$ and $\check{o} \leftrightarrow \check{o}$, so that, for A_{12},

$$\begin{array}{ccc}
(e,e,e) & \leftrightarrow & (e,e,e) \\
(o,o,o) & \leftrightarrow & (\check{e},\check{e},\check{e}) \\
(\check{e},\check{e},e) & \leftrightarrow & (o,o,e) \\
(\check{o},\check{o},o) & \leftrightarrow & (\check{o},\check{o},\check{e})
\end{array}$$

This proves that A_{12} indeed converts Λ_2^+ into D_{12}^+ and vice versa.

It is interesting that basically everything is done by rotating vectors with the help of an Hadamard matrix. An h-dimensional Hadamard matrix is a matrix formed by elements of h h-dimensional vectors such that these elements can be only 1 or -1 and all these h vectors are mutually orthogonal. It is an open problem if there exists an Hadamard matrix in every dimension $4k$. For dimensions which are powers of 2 there exists a so-called Sylvester construction of Hadamard matrices, which is built upon the following fact: if $h \times h$ matrix H_h is an Hadamard matrix, then the following $2h \times 2h$ matrix will also be an Hadamard matrix:

$$H_{2h} = \begin{pmatrix} H_h & H_h \\ H_h & -H_h \end{pmatrix} \tag{5.16}$$

Our H_4 matrix which was used to rotate lattice is of this very type: it is built by applying this construction two times to 1×1 matrix

$$(1) \tag{5.17}$$

It turned out that for two other lattices, namely, with $p = 3$ and $p = 4$, everything can be done also with the help of an Hadamard matrix. However in these cases we need a 16×16 matrix H_{16} rather then a 4×4 Hadamard matrix H_4. H_{16} is obtained from H_4 again by applying the above construction two times. We do not present here its explicit form, because it is too bulky, and it is straightforward to obtain it.

The proof for $p = 3$ case is then as follows: after dividing out the $\theta[0]$ common part and performing summation over epsilons in a way analogous to $p = 2$ case we obtain a sum over g copies of some lattice Λ_3^+. Λ_3 is a subset of integer lattice in 14-dimensional space with coordinates

$$\left(n_1^1, n_2^1, n_{12}^1, n_3^1, n_{13}^1, n_{23}^1, n_{123}^1, n_1^2, n_2^2, n_{12}^2, n_3^2, n_{13}^2, n_{23}^2, n_{123}^2\right). \tag{5.18}$$

Notation for indexes of n is analogous to the one used in $p = 2$ case. Denote $n_1^1 + n_1^2$ by m_1, $n_2^1 + n_2^2$ by m_2 and so on. Then Λ_3 is defined by the following conditions:

$$m_1 + m_{12} + m_{13} + m_{123} \,\vdots\, 2, \tag{5.19}$$

$$m_2 + m_{12} + m_{23} + m_{123} \,\vdots\, 2, \tag{5.20}$$

$$m_3 + m_{13} + m_{23} + m_{123} \,\vdots\, 2 \tag{5.21}$$

Λ_3^+ is then obtained from Λ_3 by shifting it by vectors

$$\left(\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}\right), \tag{5.22}$$

$$\left(0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, \frac{1}{2}\right), \tag{5.23}$$

$$\left(0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \tag{5.24}$$

and then taking union of three resulting lattices with the original one.

Then the final statement is that lattice $(E_7 \oplus E_7)^+$ is turned into Λ_3^+ by rotation with orthogonal matrix $A_{16} = \frac{1}{4}H_{16}$, where H_{16} is the 16-dimensional Sylvester-type Hadamard matrix. A question may arise: how can we rotate *a priori* 14-dimensional lattices into one another using a 16×16 matrix? The answer is simple: the E_7 lattice is usually defined in

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8-dimensional space (in fact it lies in a particular hyperplane inside 8-dimensional space), and therefore $(E_7 \oplus E_7)^+$ is naturally defined in 16-dimensional space, not 14-dimensional one [20]. Therefore we can simply rotate $(E_7 \oplus E_7)^+$ with the help of A_{16}. And then it turns out that the two auxiliary coordinates (namely, the first and the ninth ones) are zero for images of all vectors of $(E_7 \oplus E_7)^+$! Therefore we can just drop them out and obtain a 14-dimensional lattice. Simple technical calculations, which we do not provide here, show that this 14-dimensional lattice is indeed Λ_3^+ (basically this calculations consist of checking that image of every basis vector can be shifted back by one of the three vectors (5.22)–(5.24) to an integer vector satisfying conditions (5.19)–(5.21)). This finishes the proof of relation (5.1) for the $p = 3$ case.

The proof for $p = 4$ is absolutely analogous to $p = 3$ one except there will be 15-dimensional space instead of 14-dimensional. It is interesting that everything is again achieved with the help of the same A_{16} matrix as in the previous case.

Now it is clear, why we chose our enumeration for lattice theta constants. It was done because functions ξ_p have natural enumeration, and functions ϑ_p correspond to them for $p = 0 \ldots 4$.

We end this section by reminding the well-known formulae for the lattice theta constants for **even** 16-dimensional lattices, i.e. for $E_8 \oplus E_8$ and D_{16}^+:

$$\vartheta_6^{(g)} = 2^{-2g} \left(\sum_e \theta^8[e] \right)^2, \tag{5.25}$$

$$\vartheta_7^{(g)} = 2^{-g} \sum_e \theta^{16}[e] \tag{5.26}$$

6 The strange lattice

In the previous section we discussed the fact that, for $p = 0 \ldots 4$, functions ϑ_p coincide (up to a simple constant factor) with the functions ξ_p. That is, the lattice theta series for lattices $\mathbb{Z}^{16}$, $\mathbb{Z}^8 \oplus E_8$, $\mathbb{Z}^4 \oplus D_{12}^+$, $\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+$ and $\mathbb{Z} \oplus A_{15}^+$ are equal to combinations of the same type of ordinary theta constants.

However, there is one more odd 16-dimensional lattice, namely $(D_8 \oplus D_8)^+$. To our surprise, the corresponding lattice theta constant, ϑ_5, behaves very different from the others.

With the help of OPSMY linear relations (4.3)–(4.6) between lattice theta constants we can express ϑ_5 through $\vartheta_0, \ldots, \vartheta_4$ for every genus $g \leq 4$. Then, knowing the formulae from the previous section, expressing $\vartheta_0, \ldots, \vartheta_4$ through the ordinary theta constants, we can do the same with ϑ_5 for all genera $g \leq 4$. It turns out that these expressions in all these genera follow one and the same pattern:

$$\vartheta_5^{(g)} \underset{\mathcal{M}}{=} 2^{-\frac{g(g-1)}{2}} \left(\prod_{i=1}^{g} (2^i - 1) \right)^{-1} G_g^{(g)}, \quad g \leq 4 \tag{6.1}$$

Thus the last lattice theta constant surprisingly coincides (again up to a constant factor, this time a little bit more sophisticated) with an expression through the ordinary theta constants of the *second* type of the two mentioned in section 3. This, however, is valid only on Jacobian locus, because for genus 4, where for the first time Jacobian locus differs from the entire Siegel half-space, the linear relation (4.6) holds only on Jacobian locus. On the entire Siegel half-space formula (6.1) would then look like

$$\vartheta_5^{(4)} = \frac{1}{2^6 \cdot 315} G_4^{(4)} + \frac{3}{7} (\vartheta_6 - \vartheta_7) = \tag{6.2}$$

$$= \frac{1}{2^6 \cdot 315} G_4^{(4)} - \frac{3}{2^8 \cdot 7} F^{(4)} \tag{6.3}$$

We could not prove the formula (6.1) by a direct method like the one used for other lattices. This is unfortunate, because starting from genus 5 there are no linear relations between lattice theta constants. Therefore a question arises: would lattice theta constant ϑ_5 continue to follow the same pattern (6.1) for genera $g \geq 5$? At the moment we cannot answer this question due to various technical difficulties. Of course, equality of type (6.1) for genus 5 cannot be valid on entire Siegel space, since the right hand side, i.e. $G_5^{(5)}$, contains square roots of monomials in theta functions, and therefore possesses singularities on the entire Siegel space. However, R. Salvati Manni in the paper [9] argued that all this singularities lie outside Jacobian locus. Thus the equality could in principle be true, if restricted on Jacobian locus, despite it would be a rather mysterious theta-constant identity. Perhaps, it can help to shed some light on the possible explicit form of Schottky identities at higher genera.

7 Relations between Grushevsky and OPSMY ansätze

In this section we briefly discuss relation between the two ansätze for NSR measures.

If we substitute expressions (5.1) for the lattice theta constants into the formulae (4.13)–(4.16) for OPSMY ansatz for genera g from 1 to 4, we straightforwardly obtain formulae (3.7)–(3.10) for Grushevsky ansatz. Therefore for $g \leq 4$ both ansätze coincide, which is in perfect agreement with uniqueness properties of OPSMY and Grushevsky ansätze.

For genus $g = 5$ case it is difficult to compare these two ansätze because we do not know how ϑ_5 relates to ordinary theta constants. We can do this if we assume that (6.1) continues to hold beyond $g \leq 4$, i.e. that

$$\vartheta_5^{(5)} \underset{\mathcal{M}}{\overset{?}{=}} 2^{-\frac{5(5-1)}{2}} \left(\prod_{i=1}^{5} (2^i - 1) \right)^{-1} G_5^{(5)} \tag{7.1}$$

If we make this assumption, then we again can substitute all expressions for lattice theta constants into expression (4.17) for OPSMY ansatz and see that the resulting formula is the same that (3.11) for Grushevsky ansatz. Additional parts, entering modified expressions for ansätze that come from subtracting cosmological constant divided by the number of characteristics, are then also equal. Thus, if the function ϑ_5 continues to follow for $g = 5$ the same pattern it followed for $g \leq 4$ – despite $G_5^{(5)}$ contains square roots! – then Grushevsky and OPSMY ansätze coincide at genus 5 too.

8 Conclusion

In this paper we discussed some relations between lattice and Riemann theta constants and ansätze for superstring measures which are written in terms of them. We presented explicit formulae expressing lattice theta constants for eight 16-dimensional self-dual lattices through Riemann theta constants. This was then used to explicitly show that Grushevsky and OPSMY ansätze coincide for $g \leq 4$, as it was originally predicted. However, already for $g = 5$ there are difficulties to see if the ansätze remain the same. This is related to the strange behaviour of the theta constant, associated with one of the lattices, $(D_8 \oplus D_8)^+$. It is a very interesting problem to see exactly, how it can be expressed through Riemann theta constants for genera $g \geq 5$. Coincidence of ansätze implies a very elegant – but also extremely surprising — formula for this lattice theta-constant, but we did not find an equally nice way to prove (or to reject) it.

At the same time, despite the beauties, once again found in this paper in the world of the modular forms, it seems that hypothesis that NSR measures can be always expressed in their terms is overoptimistic. Most probably, like Mumford measure $d\mu$ itself, the ratio of $d\mu[e]/d\mu$ is going to be a function on moduli space without a natural non-singular continuation to entire Siegel half-space. At best it can contain Schottky-related forms in denominator, like it happens to $d\mu$ at genus $g = 4$ [6]. To find these ratios for genera $g \geq 5$ one should apply or invent some other technique.

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